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Efficiency Modes Analysis of Structure-Control Systems

Hayrani Öz*
Ohio State University, Columbus, Ohio 43210

An efficiency eigenvalue problem is defined associated with the structure-control system efficiency previously defined by the author. The eigenvalue problem leads to the definition of controller efficiency modes alongside the familiar structural modes. The control-structure interaction is characterized through a spectral decomposition of unique control power matrices that establishes the power interactions between the structural modes and the controller efficiency modes. In a related manner, structural mode efficiencies and controller mode efficiencies of the system are defined. A relationship between the initial states and the control system design is also identified. A significant feature of the approach is that the behavior of a lower order control model is established directly relative to the behavior of a (theoretically) ∞-dimensional entire truth model based on computations performed only in the lower order control-design space without any knowledge of the truncated dynamics. The efficiency approach as proposed in this paper is not a technique by which control laws are designed. Instead, it is a necessary common denominator engineering procedure whereby control laws already designed elsewhere via any control theory are analyzed and evaluated from the perspective of the power efficiency of the structure-control system. The efficiency modal analysis is demonstrated on representative space structures that employ linear quadratic regulator control designs.

I. Introduction

THE concept of efficiency of an engineering system is a time-proven design and analysis tool in many disciplines. A structure-control system (SCS) should not pose an exception to this practice. The concept of power efficiency as a nondimensional measure of structure-control system evaluation has been introduced recently in Ref. 1. Physically, efficiency as it pertains to an SCS is defined as the fraction of total control power absorbed usefully by the SCS toward accomplishing the control objectives. The remaining power simply represents waste of resources culminating in undesirable behavior of the system. The premise of the introduction of efficiency is that within the given constraints, SCSs that are most power efficient in satisfying the control objectives must be considered for implementation. The efficiencies for SCSs, are given in Ref. 1 for partial differential and finite element models (FEMs) as surrogates for the distributed-parameter systems (DPSs). With the efficiency definitions presented in Ref. 1, a range of issues for an SCS such as control spillover, model and controller order determination, actuator distribution, effect of structural parameters, mode shapes, closed-loop eigenvalue spectrum, and the initial disturbances can be studied. SCSs can be analyzed and synthesized from an efficiency point of view. Reference 1 presented results studying the efficiencies of various linear quadratic regulator (LQR) control designs for the ACOSS-4 (Active Control of Space Structures) structure serving as an example to study the issues relating to control of large space structures.

A conceptually and computationally significant feature of the efficiency approach is that the behavior of a low-order $2n$-dimensional control model is determined directly relative to the behavior of the (theoretically ∞-dimensional) entire truth model (TM) of the structure based on computations performed only in the $2n$-dimensional space of the control model. This is in stark contrast to common practices to date by which the performance of a low-order control model can only be established relative to an intermediate evaluation model (EM) obtained by an open-loop model reduction approach from the truth model. The truth model is either an ∞-dimensional distributed-parameter model of the structure or a very high order surrogate FEM of the structure involving hundreds or thousands of degrees of freedom. For example, in the well-known component cost analysis[2] or optimal projection[3] techniques, the performances of the reduced-order controller designs are established only relative to an evaluation model (EM) restricted to a Riccati-solvable dimension that is more often than not a much smaller model than the TM. Thus in the current approaches the question of what happens in the TM beyond the EM is left unanswered, and it is only hoped that no anomalies will be encountered when control is implemented on the TM. On the other hand, the efficiency approach introduces a framework in which the need for an intermediate EM is obviated and the TM and control model are directly related to each other without going beyond the dimension of the control model and without requiring any knowledge of the truncated dynamics beyond the control model.

The efficiency approach as discussed in this paper is not a technique by which control laws are to be designed. Instead, it is a necessary common-denominator engineering procedure by which controllers already designed elsewhere via any control theory [such as LQR, linear quadratic Gaussian (LQG), controller reduction via component cost analysis, optimal projection method, balanced approaches, etc.,[2–4]] are to be studied and evaluated from the perspective of power efficiency of the SCS. Following the introduction of the efficiency concept in Ref. 1, in this paper we introduce the concept of efficiency modes of an SCS. The efficiency modes are distinct from the natural modes of the structure; they are derived from an eigenvalue problem based on power efficiency quotients. Based on this, the main objective of the paper is to offer a host of new, physically meaningful information about the inner workings of an SCS from the efficiency perspective. To this end, the characteristic efficiencies of SCS are identified and the structural mode efficiencies and controller mode efficiencies are defined. The efficiency analysis also leads to identification of a relationship between the initial state disturbance and the control system design, which becomes an issue of efficient use of control power. By using the spectral properties of an efficiency quotient, power decomposition matrices are introduced to exhibit the interaction of structural modes and the controller modes. Essentially, the results yield an internal description of the control-structure interaction not offered in the literature heretofore. The concepts presented in the paper are intended to be the building blocks of a longer term objective of establishing an efficiency methodology for SCSs.

We illustrate the efficiency modes analysis by seven examples on the ACOSS-4 and ACOSS-6 structural models of the Charles Stark Draper Laboratory (CSDL) that employ control laws designed by

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*Professor, Department of Aerospace Engineering, Applied Mechanics and Aviation. Member AIAA.

the LQR theory. In particular, to underline the capability of the concepts, one of the examples demonstrates the efficiency modes analysis for a 588-state TM with a given 16th-order LQR control design based on computations only in the 16-dimensional design state space.

II. Control Power Matrices and Power Efficiency of SCS

For completeness, in this section we summarize the basic concepts of efficiency and the unique quadratic power functionals as introduced in Ref. 1. Consider the Nth order TM of the structural dynamic system, as a surrogate to the ∞-dimensional DPS, given by

$$M\ddot{q} + Kq = Q \tag{1}$$

where q and Q are the generalized coordinates and generalized input vectors and M and K are positive definite and positive semidefinite mass and stiffness matrices, respectively. Noting the transformation from a physical m control inputs vector F to the generalized inputs Q as $Q = DF$ and the modal properties of the system, Eq. (1) can be replaced by the modal system

$$\begin{gathered}\ddot{\xi}_N + [\omega^2]_N \xi_N = B_N F,\ B_N = E_N^T D \\ E_N^T M E_N = 1,\ E_N^T K E_N = [\omega^2]_N\end{gathered} \tag{2}$$

where ξ_N is the N-dimensional TM modal displacements vector of Eq. (1). Typically, we shall assume that the TM [Eq. (1)] represents a first-cut very high order FEM model of the structural system, where N can range into hundreds or thousands. The typical procedure in designing structural control systems at this point is to obtain a lower order structural model from the TM by an open-loop model reduction procedure. The resulting model is known as the EM, the size of which is dictated by the Riccati solvability requirement for the control approaches, which are based subsequently on this EM. The issues of whether to design a full-order or reduced-order controller are all addressed at this point from the perspective of the EM (e.g., see Ref. 2). The performances of the resulting controllers, which are further desired to be smaller than the size of the EM, are only valid relative to this EM. In this process, the evaluation of the performance of the control model relative to the TM is ignored. This is due to computational difficulties (to begin with, the TM is deemed to be beyond Riccati capabilities) and more importantly due to a lack of physical framework that can bring the TM into the controller evaluation process. Yet, the fact remains that the desired control model is implemented on the TM, and therefore, its performance must be ultimately evaluated based on the TM. With the efficiency concepts an intermediate EM is no longer needed. More realistically, instead, the performance of the desired low-order control model is related directly to the TM without any penalty on the control-related computations, as will be obvious in the following.

If the control model is $2n$ dimensional where $2n$ may be much smaller than the dimension $2N$ of the TM in a state space, we introduce the $2n$-dimensional modal-state vector for the desired control design model with ($2n < 2N$):

$$x = [\xi_1 \dot{\xi}_1 \xi_2 \dot{\xi}_2 \cdots \xi_n \dot{\xi}_n]^T \tag{3}$$

Hence, we have the truncated modal-state representation of Eqs. (1) and (2) as a low-order $2n$-dimensional control design model in which b_r is the rth row vector of B_N:

$$\begin{gathered}\dot{x} = Ax + BF \\ A = \text{block-dia}\begin{bmatrix} 0 & 1 \\ -\omega_r^2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0b_1^T 0b_2^T \cdots 0b_n^T \end{bmatrix}^T \\ r = 1, 2, \ldots, n < N\end{gathered} \tag{4}$$

The control powers S^R, S_C^M, and S^* defined in Ref. 1 are, with some modification in notation,

$$2S^R = \int F^T R^R F \,\mathrm{d}t, \qquad R^R = D^T M^{-1} D \tag{5}$$

$$2S_C^M = \int F^T R_C^M F \,\mathrm{d}t, \qquad R_C^M = D^T E_n E_n^T D \tag{6}$$

$$2S^* = \int Q^{*^T} R^* Q^* \,\mathrm{d}t, \qquad R^* = M^{-1} \tag{7}$$

where E_n is the n-mode partition of the modal matrix corresponding to the $(2n)$th order control design model, Eq. (4). For a stable system, with the $(2n)$th order initial state x_0, Eqs. (5–7) have the solutions

$$2S^R = x_0^T P^R x_0, \qquad 2S_C^M = x_0^T P_C^M x_0, \qquad 2S^* = x_0^T P^* x_0 \tag{8}$$

for which the $(2n)$th-order state feedback controls designed for the system of Eq. (4) are

$$F = Gx \tag{9}$$

and P^R, P_C^M, and P^* are positive definite genuine control power matrices, as shown in Refs. 5–7. Instead of the static state feedback assumed in Eq. (9), one can also use a dynamic state feedback control law. To elaborate on this, we denote the structural states by x_s and the compensator states by x_c and write $\dot{x}_s = A_s x_s + B_s F$ for the structural dynamics and assume $\dot{x}_c = A_c x_c + B_c F + K_c y$ for the compensator dynamics. The structural output and the compensator output are given by $y = C_s x_s$ and $F = G_c x_c + H_c y$. The assumed compensator equations can accommodate both dynamic state-estimate feedback and static direct-output feedback if either one is used for control. If a state estimator is used, the possibility of observation spillover instability arises. This, however, is an issue with the control design phase and there are various approaches available in the literature such as temporal or spatial modal filtering virtually eliminating the possibility of spillover instability (see e.g., Refs. 8–9 and the references therein). Hence, we shall assume that the nominal system of the reduced-order control design model with the particular compensator design to be analyzed for efficiency has been designed such that possibility of spillover instability no longer exists in the residual dynamics. Thus introducing the total state vector $x = [x_s^T x_c^T]^T$, the structural and compensator dynamics can be aggregated into the form $x = Ax + BF$ and $F = [H_c C_s \;\; G_c]x = Gx$, which are both symbolically equivalent to Eqs. (4) and (9). It follows that the validity of the concepts and definitions introduced in this paper via the symbolism of Eqs. (4) and (9) will not be altered whether one assumes static or dynamic state feedback. Therefore, we opt for simplicity and consider the feedback law of Eq. (9) in the sequel without loss of generality. Efficiency of SCSs with dynamic compensators is presented in Ref. 10.

With stabilizing gains G, the control power matrices P^R and P_C^M are obtained by solving the $(2n)$th-order Lyapunov equations

$$A_{\mathrm{CL}}^T P^R + P^R A_{\mathrm{CL}} + G^T R^R G = 0, \qquad A_{\mathrm{CL}} = A + BG \tag{10}$$

$$A_{\mathrm{CL}}^T P_C^M + P_C^M A_{\mathrm{CL}} + G^T R_C^M G = 0 \tag{11}$$

and a closed-form solution for P^* exists as given in Refs. 5–7. The matrix G is the control gain matrix for a finite number of spatially discrete m inputs, available from any control design technique such as the LQR theory.

The control power S^* and the generalized input Q^* in Eq. (7) require some elaboration. The generalized inputs vector Q^* is given by

$$\begin{gathered}Q^* = ME_n f^*(t), \\ f^* = [f_1^*(t) \cdots f_r^*(t) \cdots f_n^*(t)]^T\end{gathered} \tag{12a}$$

where $f^*(t)$ is the n-dimensional modal input vector having the feedback form

$$\begin{gathered}f_r^* = g_{r1}^* \xi_r^* + g_{r2}^* \dot{\xi}_r^*, \qquad g_{r1}^* = \omega_r^2 - \lambda_{r1}\lambda_{r2} \\ g_{r2}^* = -(\lambda_{r1} + \lambda_{r2}), \qquad r = 1, \ldots, n\end{gathered} \tag{12b}$$

where λ_{r1} and λ_{r2} are the rth closed-loop eigenvalues obtained with the gain matrix G of Eq. (9) used in the $(2n)$th-order control design model (4) with m spatially discrete (typically point) inputs F. The $(2n)$th-order control law described by Eqs. (12) acts on the full-order TM equation (1) by virtue of the appearance of the TM mass matrix in Eq. (12a). Physically, the control design described by Eqs. (12) represents a globally optimal spatially continuous input field on the ∞-dimensional DPS and, hence, on its surrogate finite N-dimensional TM, Eq. (1). Note that in Eq. (12b), the control law is obtained for a low-order $2n$-dimensional control design model. The control laws described by Eqs. (9) and (12) have the same closed-loop eigenvalue spectrum, although they represent different input fields. Such control systems are referred to as dynamically similar[11]. The Frobenius norm of the gain matrix of Eqs. (12) is globally minimum in comparison to the norm of G in Eq. (9). The control law described by Eqs. (12) is the natural control for the DPS, also known as independent modal space control (IMSC)[6,12]. One can easily verify that the spatially continuous control given by Eq. (12a) has no control spillover beyond the control design model[1]. The natural control then embodies an ideal globally optimal control concept with 100% efficiency for the complete ∞-dimensional DPS, or the TM, regardless of the size of the control-design model.

We now return to the control power expressions in Eqs. (5–7). By virtue of the orthogonality relationships in Eq. (2) one has $M^{-1} = E_N E_N^T$; therefore, S^R represents the total real control power expended on the ∞-dimensional DPS or on the TM by the m spatially discontinuous inputs F designed based on the low-order control-design model, Eq. (4). Hence the real power functional S^R on the TM is directly related to the control-design model indicating the behavior of the TM when the control model is implemented on the TM. On the other hand, noting the form of R_C^M in Eq. (6), S_C^M represents the projection of S^R on the $2n$-dimensional modal control-design space of the structure; therefore $S_C^M \leq S^R$. Here S^* is the minimum control power that one would expect to expend for the same control task on the DPS if a dynamically similar spatially continuous input field were to be used as discussed previously. Hence, the following properties hold:

$$S^R \geq S_C^M, \qquad S^R \geq S^*, \qquad S_U^M = S^R - S_C^M \tag{13}$$

where S_U^M is the control power spillover to the residual dynamics of the system beyond the $2n$-dimensional control-design space. Noting Eqs. (13) and (8), we also have

$$2S_U^M = x_0^T\left(P^R - P_C^M\right)x_0 = x_0^T P_U^M x_0 \tag{14}$$

Hence, the control power wasted on the uncontrolled dynamics is readily available following the solutions of $(2n)$th-order Lyapunov equations (10) and (11) based on the $(2n)$th-order controlled dynamics alone; no explicit knowledge of the residual modes is required. Finally, note that S^R, S_C^M, and S^* are defined irrespective of the specific nature of the control-design model, that is, Eq. (4) does not have to be a modal space of the structure. All that is needed to obtain the control powers is the knowledge of the inputs F and Q^*. The central idea in the efficiency approach is to begin by first evaluating the control power functionals S^R, S_C^M, S^* for any given control input history regardless of the control theory used to design or compute them (e.g., even if it may be that the control inputs may have been designed by optimizing some LQR or LQG performance measure).

From the unique power functionals defined in Eqs. (5–7), the following power efficiency quotients and the associated power spillover quotients are defined for the SCS[1]

$$e^*\% = (S^*/S^R) \times 100 \leq 100\%, \qquad e\% = \left(S_C^M/S^R\right) \times 100 \leq 100\% \tag{15}$$

where e^* and e are the global and relative-model power efficiencies, respectively, and the two are related by the modal efficiency coefficient $\mu = S_C^M/S^*$, $e = \mu e^*$. Accordingly the global and relative-model power spillover quotients are also defined as

$$\text{sq}^* = S_U^M/S^* = 1/e^* - \mu, \qquad \text{sq} = S_U^M/S^R = (1 - e) \tag{16}$$

Using Eqs. (8) in Eq. (15),

$$e = x_0^T P_C^M x_0 / x_0^T P^R x_0 \tag{17a}$$

$$e^* = x_0^T P^* x_0 / x_0^T P^R x_0 \tag{17b}$$

and sq and sq* follow similarly:

$$\text{sq} = x_0^T\left(P^R - P_C^M\right)x_0 / x_0^T P^R x_0 \tag{18a}$$

$$\text{sq}^* = x_0^T\left(P^R - P_C^M\right)x_0 / x_0^T P^* x_0 \tag{18b}$$

Most importantly, it must be noted that the weighting matrices R in Eqs. (5–7) and, hence, the control powers are uniquely defined for the purpose of structural control system efficiency. It is also only for those particular R matrices that the connection between the control model and the TM is established uniquely and directly. Having obtained the power matrices, one can now compute e, e^*, sq, and sq* for any given feedback control law according to their definitions in Eqs. (15–18).

III. Efficiency Eigenvalue Problem, Controller Efficiency Modes, and Principal Controller Efficiency Axes

With the symmetric, positive definite power matrices, an efficiency quotient is also a Rayleigh quotient, and an efficiency eigenvalue problem can be defined to analyze the SCS. To this end, in the following, we only need to define a numerator and a denominator power matrix without necessarily referring to any of the quotients e, e^*, sq*, and sq and use e as a generic symbol to represent any of them. Hence, consider the general form of an efficiency quotient

$$e = x_0^T P_N x_0 / x_0^T P_D x_0, \qquad P_N, P_D > 0 \tag{19}$$

where subscripts N and D denote numerator and denominator power matrices. For a given SCS both P_N and P_D are available from Eqs. (10) and (11) and Ref. 7 if $P_N = P^*$.

Next, we consider the eigenvalue problem associated with the symmetric positive definite power matrices

$$P_N t_r = \lambda_r P_D t_r, \qquad T = [t_1 t_2 \cdots t_{2n}], \qquad r = 1, 2, \ldots, 2n \tag{20}$$

where t_r is a real eigenvector with positive real eigenvalue λ_r, T is the associated modal matrix normalized to satisfy

$$T^T P_D T = I, \qquad T^T P_N T = \Lambda, \qquad \Lambda = \text{diag}[\lambda_1 \lambda_2 \cdots \lambda_{2n}] \tag{21}$$

and I is a $2n \times 2n$ identity matrix. Introducing the new modal transformation

$$x = T\epsilon \tag{22}$$

and using the orthonormality conditions (21), we obtain the quotient e in the form

$$e = \epsilon_0^T \Lambda \epsilon_0 / \epsilon_0^T \epsilon_0 \tag{23}$$

We shall refer to the new states ϵ as the efficiency states (ϵ-states). In terms of the efficiency states, the quotient (23) can be expanded in the form

$$e = \sum_{i=1}^{2n} c_i^2 \lambda_i, \qquad c_i^2 = \epsilon_{0i}^2 / \epsilon_0^T \epsilon_0, \qquad 0 \leq c_i \leq 1 \tag{24}$$

in which we shall refer to c_i^2 and λ_i as the ith efficiency coefficient and the ith characteristic efficiency, respectively. Next, introducing the inverse transformation of (22)

$$\epsilon = Lx, \qquad L = T^{-1} = T^T P_D \tag{25}$$

into Eq. (23) and considering the original form of the efficiency quotient in terms of structural states x [Eq. (19)], we obtain the following spectral forms of the power matrices:

$$P_N = L^T \Lambda L, \qquad P_D = L^T L \tag{26}$$

The modal-state dynamics [Eq. (4)] can also be represented in terms of the efficiency states by using transformation (22):

$$\dot{\epsilon} = A^e\epsilon + B^eF, \qquad A^e = LAT, \qquad B^e = LB \tag{27}$$

The feedback control law for F in Eq. (9) becomes

$$F = G^e\epsilon, \qquad G^e = GT \tag{28}$$

In terms of ϵ states the closed-loop system becomes

$$\dot{\epsilon} = A^e_{\text{CL}}\epsilon, \qquad A^e_{\text{CL}} = A^e + B^eG^e = LA_{\text{CL}}T \tag{29}$$

By using Eqs. (26) and (29) in Eqs. (10) and (11), the Lyapunov equations associated with the ϵ-state space representation are obtained as

$$\begin{aligned} A^{eT}_{\text{CL}}\Lambda + \Lambda A^e_{\text{CL}} + G^{eT}R^M_C G^e = 0, \\ A^{eT}_{\text{CL}}I + IA^e_{\text{CL}} + G^{eT}R^RG^e = 0 \end{aligned} \tag{30}$$

Hence, Λ is recognized as the numerator power matrix and the identity matrix I is the denominator power matrix associated with the definition of an efficiency quotient in the ϵ-state space. Returning to the new modal matrix T, just as the structural modal matrix E_N diagonalizes the structural mass and stiffness matrices according to Eqs. (2), the modal matrix T diagonalizes similarly both of the control power matrices P_N and P_D according to Eq. (21). By analogy, we shall refer to the T matrix as the controller efficiency modal matrix and to its columns t_r as the efficiency modes of the control system.

Next the efficiency equation (24) can be written in the form

$$\sum_{i=1}^{2n} c_i^2\frac{\lambda_i}{e} = 1, \qquad c_i = \frac{\epsilon_{i0}}{|\epsilon_0|} = \frac{L_i^Tx_0}{|\epsilon_0|} \tag{31}$$

where L_i is the ith row vector of L. Equation (31) represents the equation of an ellipsoid in the $2n$-dimensional design space with principal axes of length $\sqrt{e/\lambda_i}$ $(i = 1, \ldots, 2n)$ and c_i is the coordinate along the ith principal axis of all initial disturbances that yield a specific efficiency e. With respect to the efficiency eigenvalue problem associated with the original modal coordinates x [Eq. (19)], the ith principal axis of the ellipsoid (31) is given by the ith column t_i of the transformation matrix T [Eq. (22)], which is the modal matrix of quotient (19). We note that the directions t_i are orthonormal, not in the usual sense, but with respect to the denominator control power matrix P_D, as given in Eq. (21).

Because $0 < e \leq 1$ (excluding the case when $e = \text{sq}^*$), different initial conditions x_0 yielding different efficiencies simply results in rescaling of the lengths of the principal axes of the ellipsoid (31), the largest length in the ith direction given by $\sqrt{1/\lambda_i}$. The largest and the smallest possible lengths of the principal axes are then $\sqrt{1/\lambda_{\min}}$ and $\sqrt{1/\lambda_{\max}}$, respectively. We shall refer to the ellipsoid described by Eq. (31) as the efficiency ellipsoid and, alternately, refer to the efficiency modal vectors t_i as the principal controller efficiency axes. We note that the efficiency modes, characteristic efficiencies, and shape of the efficiency ellipsoid are completely and only defined by the particular SCS design represented by the matrices G, A, and B in Eqs. (9) and (4). Initial disturbances x_0 then determine the size of the efficiency ellipsoid acting merely as a scaling factor to yield a specific power efficiency for the SCS.

Because a quotient e in the form of Eq. (19) represents a Rayleigh quotient and since $0 < e \leq 1$, we observe the following properties for efficiencies:

1) The characteristic efficiencies λ_i are bounded by $0 \leq \lambda_i \leq 1$ for e, e^*, and sq; for the quotient sq*, $\lambda_i \geq 0$.

2) Efficiencies have stationary values at λ_r for initial states $x_0 = t_r, r = 1, 2, \ldots, 2n$. Specifically, the maximum efficiency an SCS can achieve is $e_{\max} = \lambda_{\max}$, and will occur if $x_0 = t_{\max}$, i.e., if the initial state coincides with the eigenvector (controller efficiency mode) associated with $\lambda_{\max}$, i.e., if x_0 is completely aligned with the direction of the controller principal axis along $t_{\max}$. Similarly, the lower bound of the efficiency of the system is given by $e_{\min} = \lambda_{\min}$, corresponding to $x_0 = t_{\min}$. Here $\lambda_{\min}$ will be referred to as the fundamental efficiency of the SCS.

In fact, if quotient (19) is the relative model spillover quotient defined by Eq. (18a), it then represents the percentage of inefficiency (fraction of control power wasted on truncated dynamics) of the control system. Denoting the eigenvalues of the relative model efficiency quotient e defined in Eq. (17a) by λ^e and the eigenvalues of the spillover quotient sq defined in Eq. (18a) by λ^s and noting that $P^M_C = P^R - P^M_U$, the specific form of Eq. (20) for λ^e_r yields

$$\left[\left(1 - \lambda^e_r\right)P^R - P^M_U\right]t_r = 0, \qquad r = 1, 2, \ldots, 2n$$

which constitutes the eigenvalue problem for the relative model spillover quotient sq. Hence, the eigenvalues of the spillover quotient sq (or equivalently, controller power inefficiency) are $\lambda^s_r = 1 - \lambda^e_r, r = 1, 2, \ldots, 2n$, with the same eigenvectors t_r as that of the relative model efficiency e. Similar to expansion (24) for the model spillover quotient we can write the expansion

$$\text{sq} = \sum_{i=1}^{2n} c_i^2\lambda^s_i = \sum_{i=1}^{2n} c_i^2(1 - \lambda^e_i) = \sum_{i=1}^{2n} c_i^2 - \sum_{i=1}^{2n} c_i^2\lambda^e_i$$

which yields, after recognizing $\sum_{i=1}^{2n} c_i^2 = 1$, $\text{sq} = 1 - e$, as is given in Eq. (16).

Note that although there may be infinitely many truncated modes in the system, what happens from the point of view of control power efficiency is described completely in the $2n$-dimensional control design space spanned by the controller efficiency modes t_r, $r = 1, 2, \ldots, 2n$.

IV. Principal (Controller Mode) Efficiency and Initial States

The ultimate efficiency of a control system is dictated together with the structure and control design represented by the matrices A, B, and G and the initial disturbance x_0. To see how the control design and the initial states interact, consider the expansion of an efficiency quotient given by Eq. (24) and write

$$e = \sum_{i=1}^{2n} e_i, \qquad e_i = c_i^2\lambda_i \tag{32}$$

where e_i represents the efficiency of the controller along the ith principal controller efficiency axis t_i. We shall refer to e_i as the ith principal efficiency or the ith controller mode efficiency. For a given SCS, expressing the functional dependencies of the quantities involved in the principal efficiencies in Eq. (32) as arguments by recognizing Eqs. (10), (11), (19), and (31), one can write

$$e_i = c_i^2\lambda_i = \frac{x_0^TL_i(A, B, G)L_i^T(A, B, G)x_0}{x_0^TL^T(A, B, G)L(A, B, G)x_0}\lambda_i(A, B, G) \tag{33}$$

and observe the effects of structure and control-design parameters on c_i, λ_i, and e_i. Common sense requires that an efficient controller must have high controller mode efficiencies e_i in the principal controller efficiency directions. Although a control design might have a large characteristic efficiency λ_i, it may not necessarily have a high principal efficiency e_i unless the projection c_i of the initial disturbance x_0 is also significant along the direction t_i of large λ_i. For a given structure, considering that both c_i and λ_i are functions of control-design matrices G and B as implied by Eq. (33), it may, therefore, be the control design itself that may either enhance or degrade its principal (controller mode) efficiency.

Traditionally, in the design of a control system (such as in the LQR-type designs) there is no avenue to bring in the initial-state information to the computation of the control gain matrices. After having obtained a so-called optimal controller via LQR algorithms, one does not really know how the control system will interact with any initial disturbance until a simulation is done. It is possible that an optimal controller (in the sense of the theory used to design it) may be an extremely inefficient controller if it does not receive significant projections of the initial disturbance x_0 along its principal controller efficiency axes associated with high λ_i, as will be illustrated in Sec. VI.

On the other hand, the efficiency mode analysis of an SCS clearly uncovers the interaction between the controller design and the initial disturbances x_0. In fact, this interaction is a matter of efficient use of the control power. For any initial disturbance x_0, after identifying the modal matrix T and λ_i, one can readily compute the projections c_i via Eq. (31) to examine how the initial state pairs up with the respective characteristic efficiencies λ_i and obtain an a priori information about the controller efficiency performance. Certainly, the observation of the link between the controller design and the initial disturbance is a constructive one so that given an initial disturbance and a structure, one of the objectives might be to design a control system with significant projections c_i along significant λ_i to have high principal (controller mode) efficiencies e_i.

V. Decomposition of Efficiency Quotients

In the efficiency modes analysis of a given $(2n)$th-order SCS design model, we identified $2n$-controller efficiency modes that led to the efficiency state (ϵ-state) description of the original design model in the structural modal space (x-state). From an interdisciplinary perspective, how the controller efficiency modes and the structural modes relate to each other would be of interest. The conjecture is that poor alignment of the controller modal directions with the structural modal directions will result in controllers not channeling the control power efficiently to the modeled structural modes and wasting it to the unmodeled-structure, truncated dynamics. Conversely, favorable alignment of the controller modes and the modeled structural modes will lead to best possible use of control power in satisfying control objectives. The interaction between the structural modes and controller modes can be described and quantified by decomposition of efficiency quotients by defining decomposition matrices for the numerator and denominator control power matrices. These decompositions, in turn, eventually can be used to correlate the structural and controller modes.

Associated with any of the quantities e, e^*, sq, and sq*, we define $2n$-dimensional numerator and denominator matrices $[N]$ and $[D]$ with elements

$$[N] = N_{ij} = \left[X_0 L_i L_i^T \lambda_i\right]_{jj}, \qquad [D] = D_{ij} = \left[X_0 L_i L_i^T\right]_{jj}$$
$$i, j = 1, 2, \ldots, 2n \tag{34}$$

where $X_0 = x_0 x_0^T$ is the initial-condition covariance matrix. Next, by forming the total sums N and D, one can verify that

$$N = \sum_i^{2n} \sum_j^{2n} N_{ij} = x_0^T P_N x_0$$
$$D = \sum_i^{2n} \sum_j^{2n} D_{ij} = x_0^T P_D x_0 \tag{35a}$$

$$e = N/D, \qquad e = \{e, e^*, \text{sq}, \text{sq}^*\} \tag{35b}$$

We shall refer to matrices $[N]$ and $[D]$ as control power decomposition matrices. Each element N_{ij} or D_{ij} represents a power influence coefficient indicating the power consumption of the jth structural mode x_j from the ith controller mode ϵ_i for the respective numerator and denominator terms. The quotient e in terms of sums-of-power-influence coefficients as given in Eq. (35a) represents an efficiency decomposition. Although through the controller design one has the power matrices P_N and P_D available in the beginning, their representations in terms of the elements N_{ij} and D_{ij} in Eq. (34) unveils the internal structure of the power distribution associated with the controller that is only possible after the efficiency modal analysis of Sec. III is performed. Next, we define the partial sums

$$N_j = \sum_{i=1}^{2n} N_{ij}, \qquad D_j = \sum_{i=1}^{2n} D_{ij}$$
$$j = 1, 2, \ldots, 2n \tag{36}$$

as the power consumption of jth structural modal state x_j from all controller modes. Similarly, we define power consumptions of all structural modal states from the ith controller mode:

$$N_i = \sum_{j=1}^{2n} N_{ij}, \qquad D_i = \sum_{j=1}^{2n} D_{ij},$$
$$i = 1, 2, \ldots, 2n \tag{37}$$

Considering definitions (37) and the definition of the ith principal efficiency or controller mode efficiency e_i [Eq. (32)], it follows that

$$e_i = \frac{N_i}{\sum_i D_i} = \frac{N_i}{D} = c_i^2 \lambda_i, \qquad i = 1, 2, \ldots, 2n$$

From Eqs. (24) and (32), e_i is also the contribution to efficiency of the ith efficiency state ϵ_i.

Similarly, one can identify efficiency contributions with respect to the structural modal directions and define

$$e_j = \frac{N_j D + D_j N}{2D^2}, \qquad e = \sum_j^{2n} e_j \tag{38}$$

We shall refer to e_j as the jth structural mode efficiency. One should note the difference in the terminology being used that in e_j the subscript j denotes a structural mode efficiency and in e_i the subscript i denotes a controller mode efficiency. More specifically, e_j defined by Eq. (38) is a coupled structural modal efficiency in the sense that, in the definitions of N_j and D_j, terms involving products of x_{j0} with all other states appear. Hence, a structural state x_{j0} may have a high efficiency contribution e_j, not because of itself alone, but also because of the contribution of its coupling terms with other structural states. A second definition of e_j disregards all coupling terms among the structural states, and we define it as the decoupled structural modal efficiency

$$e_j = n_j / d_j, \qquad n_j = \bar{x}_{j0}^T P_{Njj} \bar{x}_{j0}$$
$$d_j = \bar{x}_{j0}^T P_{Djj} \bar{x}_{j0}, \qquad \bar{x}_{j0}^T = [\xi_j \dot{\xi}_j] \tag{39}$$
$$j = 1, \ldots, n$$

where P_{Njj} and P_{Djj} are the 2×2 block-diagonal partitions of the numerator and denominator power matrices, respectively. In the definition of structural modal efficiency e_j, we retain the pair of a modal displacement and its velocity as a component for obvious reasons. This is also to be observed in all other definitions. The decoupled structural modal efficiency e_j would be exactly the efficiency of the system if all initial states but the structural modal states j (displacement and velocity) were zero; hence, they indeed represent proper contributions to the overall efficiency. We also recognize that $e_j < 1$.

Next, we note that

$$N = \sum_j n_j + \sum_{i \neq j} \sum \bar{x}_{i0}^T P_{Nij} \bar{x}_{0j}$$
$$D = \sum_j d_j + \sum_{i \neq j} \sum \bar{x}_{i0}^T P_{Dij} \bar{x}_{0j} \tag{40a}$$
$$i, j = 1, 2, \ldots, n$$

where the second summations are the off-block-diagonal power cross-coupling terms. Since, these off-diagonal contributions are sign variant, one might expect them to vanish in a statistical sense if sufficient modal states are disturbed initially. Thus, we can write an expected total efficiency in terms of the block-diagonal decoupled powers as

$$E(e) \cong \sum_j^n n_j \Big/ \sum_j^n d_j \tag{40b}$$

where E denotes expectation. Further, we note that in this case $e \neq \sum_{j=1}^n e_j$. Many studies that we have conducted verify that the expectation of overall efficiency in terms of decoupled structural modal contributions is quite relevant and off-block-diagonal contributions to both the numerator and control powers can be insignificant.

In the definitions of both the coupled and decoupled structural modal efficiencies e_j one notes that the numerator and denominator power contributions N_j, D_j, n_j, an d_j can be computed exactly directly from the numerator and denominator power matrices P_N and P_D without invoking any of the results of the efficiency modes analysis. In these definitions of the exact structural modal contributions e_j via the power matrices, summation over all principal controller efficiency modes $i = 1, 2, \ldots, 2n$ is implicit. However, in reality there may be cases in which all controller modal power contributions will not be significant. Thus, the least efficient controller modes need not be taken into account in the summation over i. Therefore, it may suffice to consider only the partial sums involving only $n_p < 2n$ of the dominant controller efficiency modes in Eqs. (34) and (36) to write:

$$N_j \cong \left[X_0 \sum_{i=1}^{n_p} P_{Ni}\right]_{jj}, \qquad D_j \cong \left[X_0 \sum_{i=1}^{n_p} P_{Di}\right]_{jj} \tag{41a}$$

$$n_p < 2n, \qquad j = 1, \cdots, 2n$$

where

$$P_{Ni} = \lambda_i L_i L_i^T, \qquad P_{Di} = L_i L_i^T \tag{41b}$$

are controller power spectral components of the power matrices available only after an efficiency modal analysis is performed. Similarly, for approximations to n_j and d_j defined in Eq. (39) one can use the truncated power sums

$$P_{Njj} \cong \sum_i^{n_p} [P_{Ni}]_{jj}, \qquad P_{Djj} \cong \sum_i^{n_p} [P_{Di}]_{jj} \tag{41c}$$

$$j = 1, 2, \ldots, 2n$$

By utilizing the structural mode efficiencies e_j, $j = 1, 2, \ldots, 2n$ (model indices), and the (principal) controller mode efficiencies e_i, $i = 1, 2, 3, \ldots, n_p \leq 2n$ (controller indices), one is now in a position to propose model/controller reduction criteria for the SCS utilizing the decomposition information defined in this section. Possible model/controller reduction criteria will have to judge the significance of (yet to be defined) correlations between the structural modes and the controller efficiency modes based on the elements of the efficiency decomposition matrices $[N_{ij}]$ and $[D_{ij}]$, along with the definitions of e_j and e_i. Because both structural modal information and controller modal information are involved in all the developments presented heretofore, a two-way closed-loop model/controller reduction method may be possible by truncating simultaneously both the structural modes and the controller modes. The model/controller reduction via the efficiency concept is beyond the scope of this paper, and the preliminary ideas are discussed in Ref. 13–14. The topic is still under current research.

VI. Illustrative Examples

ACOSS-FOUR Structure

The tetrahedral structure shown in Fig. 1 is known as the ACOSS-FOUR model. The structure is subjected to a unit initial displacement in the x direction at node 2. The control configuration has six inputs located at the pods of the structure, two on each pod. The design objective is to control the x-y plane motion of the vertex, representing a line of sight error control. The same structure was also studied in ref. 1. The controls were designed by using the LQR theory to optimize the following control design performance index (CDPI)

$$\text{CDPI} = \frac{1}{2}\int (x^T q x + r F^T F)\,\mathrm{d}t, \qquad r = 1$$

$$q = \text{diag}\begin{bmatrix} \omega_r^2 & 0 \\ 0 & 1 \end{bmatrix}, \qquad r = 1, 2, \ldots, n \tag{42}$$

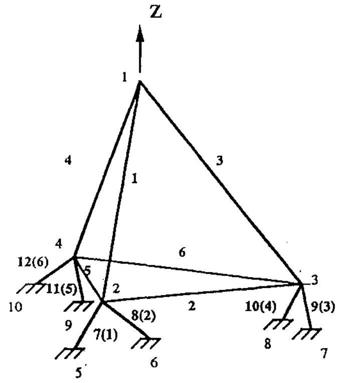

Fig. 1 ACOSS-FOUR model. Actuator numbers are in parentheses.

The finite element truth model has $N = 12$ modes, yielding a 24th-order state-space model. The natural frequencies of the structure are $\omega_r = \{1.342, 1.664, 2.891, 2.957, 3.398, 4.204, 4.662, 4.755, 8.539, 9.251, 10.285, 12.905\}$ rad/s. Examples 1–6 evaluate the given LQR designs from an efficiency perspective. Control designs based on other approaches can be efficiency analyzed similarly.

Efficiency Modes Analysis

Example 1. The LQR control design had two structural modes, modes 11 and 12, and one input (input 4) was used. Hence, $n = \{11, 12\}$, $m = 1$. The following results were obtained: $e^*\% = 0.18$, $e\% = 4.32$, $\mu = 23.50$, $S^* = 2.919$, $S_C^M = 68.591$, $S^R = 1587.285$. Low levels of power efficiencies are noted in this controller configuration.

Example 2. The same LQR control design model as in Example 1 was considered with the first four actuators as the inputs $m = 4$ (inputs 1–4). This resulted in $e^*\% = 6.20$, $e\% = 28.60$, $\mu = 4.61$, sq$^* = 11.51$, sq$\% = 71.40$, $S^* = 15.199$, $S_C^M = 70.101$, $S^R = 245.131$. In comparing these results to Example 1, the effects of the number of inputs on the efficiencies and control powers are clearly noted. The efficiency coefficients c_j^2, the characteristic efficiencies λ_i and controller modal efficiencies e_i, and the controller efficiency modal matrix T for both the global and relative model efficiencies are listed in Table 1. The results indicate that both for the relative model and global efficiencies the first controller mode listed is the dominant contributor to the controller efficiency. For both efficiencies, the initial disturbance has the largest projection along the first controller mode, although the associated characteristic efficiencies are the lowest in the spectrum. Clearly, the efficiencies are bracketed by the lowest and highest characteristic efficiencies, and the sum of the controller mode (principal) efficiencies yields the overall efficiencies. Each ith column of the T matrix listed signifies a controller mode representing an initial disturbance state that will yield the corresponding λ_i as the controller mode efficiency. Certainly, any arbitrary initial state can be expanded in terms of the controller efficiency modes characterized by the columns of T. Furthermore, we note that each row of a given column of the controller modal matrix shows the relative significance of the associated structural modal states in that controller mode. For example, in the case of global efficiency the first controller mode (first column of T) is clearly dominated by the third- and fourth-row elements, which are the states for the second (12th) structural mode. Similarly, the second controller mode is dominated by states of the first (11th) structural mode. In the case of relative model efficiency, we note that such clear dominance is not observed. In fact, both structural modes appear to be represented comparably in the first two controller modes. Simulation results of the squared line-of-sight (LOS) error for this design are shown in Fig. 2. The higher curve, designated as "Truth Model," is the simulation of the LOS error of the full-order ($N = 12$) model. The lower, dashed curve is the LOS error of the second-order control design model alone, disregarding the truncated dynamics. The effect of control power spillover because of the inefficiency of the controller as LOS response degradation is clearly depicted in Fig. 2. It is shown in Refs. 7 and 13 that an upper bound $Z_{\max}^2$ on the total squared distributed response degradation can be computed by using the formula $Z_{\max}^2 = \omega_{u,\min}^{-4}(1 - e)S^R$,

Table 1 Efficiency modes analysis for ACOSS-FOUR

Controller efficiency mode i	Global efficiency λ_i^*,%	e_i^*,%	c_i^{*2}	Relative model efficiency λ_i,%	e_i,%	c_i^2
1	5.4	5.10	0.949	28.4	26.63	0.938
2	28.0	1.03	0.037	32.3	1.38	0.043
3	5.4	0.07	0.014	32.3	0.32	0.010
4	28.0	0.00	0.000	28.4	0.27	0.010

Controller modal matrix $T = [T_{ji}]10^3$

−0.188	100.135	0.009	−1.247	0.765	99.800	−2.642	7.815
−0.104	12.805	−2.223	1030.367	100.537	27.172	1025.132	9.084
52.542	0.511	6.343	−0.093	52.315	−0.330	5.166	6.122
−81.926	−1.018	677.475	7.607	−78.940	−53.375	−6.140	675.741

i = controller efficiency mode index; j = structural mode index.

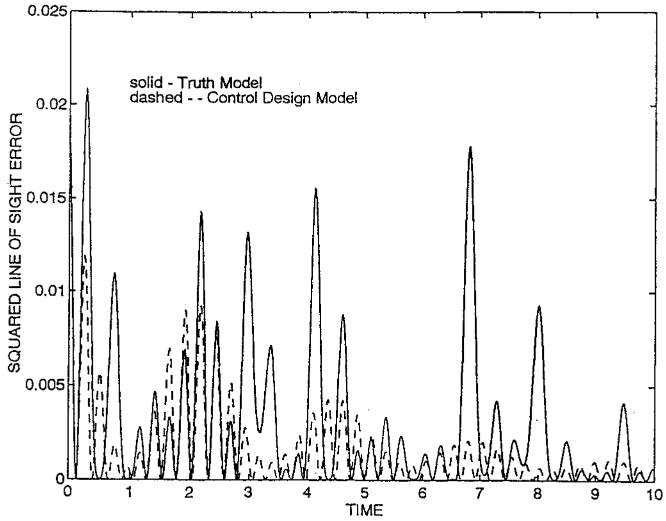

Fig. 2 ACOSS-FOUR simulations for Example 2; $n = 2$ {modes 11, 12}, $m = 4$ (1–4).

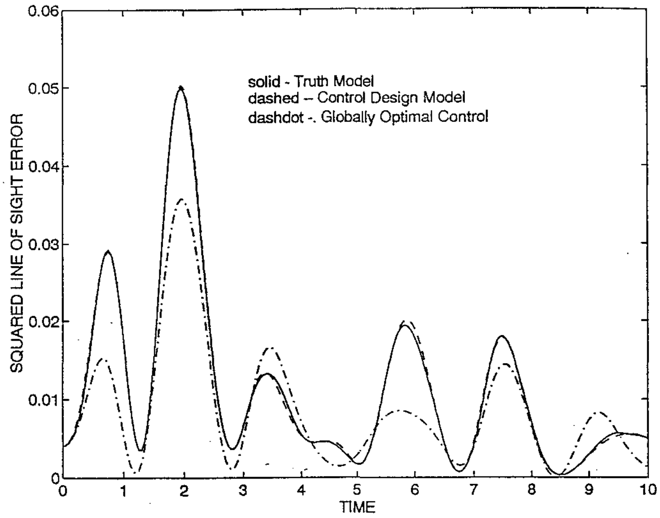

Fig. 3 ACOSS-FOUR simulations for Example 3; $n = 8$ {modes 1–8}, $m = 2$ (1, 2).

where $\omega_{u,\min}$ is the minimum natural frequency in the uncontrolled dynamics. For this design one obtains $Z^2_{\max} = 53.96$, a figure of merit that is visually manifested as the LOS response degradation in Fig. 2.

Example 3. The LQR control design model considered the first eight structural modes $n = \{1, \ldots, 8\}$ with two inputs, $m = 2$ (inputs 1 and 2). To conserve space, only the characteristic efficiencies λ_i and the corresponding controller mode efficiencies e_i are listed in Table 2. This system had $e^*\% = 8.39$, $e\% = 55.26$, $\mu = 6.59$, $\text{sq}^* = 5.34$, $\text{sq}\% = 44.74$ with $S^* = 1.667$, $S_C^M = 10.994$, $S^R = 19.896$. Figure 3 shows the simulation results for the TM, eight-order control-design model and the dynamically similar globally optimal control (IMSC). In this design there appears hardly any response degradation between the control-design model and the TM. However, suboptimal behavior of both responses relative to the globally optimal (IMSC) response is quite evident. For this design one obtains $Z^2_{\max} = 17 \times 10^{-4}$, which translates into a visually insignificant response degradation between the control-design model and the TM. Thus, a mere inspection of responses between these two models does not necessarily reveal the almost 45% power inefficiency of the controller design.

Example 4. The same system of Example 3 is analyzed for inefficiency (spillover) modes. Table 2 lists the characteristic inefficiencies λ_i^{sq} and the corresponding (principal) controller mode inefficiencies sq_i.

Effect of Initial Conditions

Example 5. For the LQR control design studied in Example 3, from Table 2 it is observed that with a mere change in the initial states x_0, this system can have a maximum global efficiency of $e^* = \lambda^*_{\max} = 53.46\%$ corresponding to the second characteristic efficiency listed. This would occur when and if initial-state disturbance x_0 coincides with the second controller efficiency mode vector $x_0 = t_2 = \{0.107 \times 10^{-5}, 0.267 \times 10^{-5}, -0.783 \times 10^{-2}, -0.114 \times 10^{-2}, -0.199 \times 10^{-1}, -0.261 \times 10^{-1}, 0.139 \times 10^{-4}, 0.259 \times 10^{-4}, 0.358, -0.240, -0.461 \times 10^{-5}, 0.116 \times 10^{-4}, 0.697 \times 10^{-7}, 0.429 \times 10^{-5}, 0.151 \times 10^{-2}, -0.371 \times 10^{-2}\}$. In contrast, the worst case would happen if $x_0 = t_{14}$, that is, the initial disturbance matches the 14th controller efficiency mode yielding the minimum global efficiency $e^*_{\min} = \lambda^*_{\min} = 0.773\%$. It must be noted that for both initial states the controller is the same and regarded optimal from the perspective of LQR theory with respect to control design performance index equation (42). But from a global efficiency point the latter so-called optimal design would not be favorable at all. For the model efficiency of this particular LQR design, we note that changes in the initial disturbance x_0 will not have a significant effect on it since all characteristic efficiencies are condensed in a narrow band $59.80 \geq \lambda^e \geq 54.05$; at most a 5.75% change in the efficiency can occur. This represents a robust design efficiency wise with respect to the initial states.

Efficiency Decomposition

Example 6. The control power decomposition matrices $[N]$ and $[D]$ are computed for the design of Example 2 for its global and relative model efficiencies. Each element $ij(i, j = 1, 4)$ of the $[N]$ and $[D]$ matrices describes the power trade between the jth structural mode and the ith principal controller mode for the respective control power matrix. The results are shown in Table 3. The ith row sum of a decomposition matrix represents the power expenditure in the ith controller mode for all structural modal states, and the jth column sum represents the power expenditure in the jth structural modal state by all controller modes. The sum of all elements of a power decomposition matrix yields the corresponding control power S. For example, each even column of the $[N]$ and

Table 2 Characteristic efficiencies and controller mode efficiencies for ACOSS-FOUR

Controller efficiency mode i	Global efficiency/inefficiency				Relative model efficiency/inefficiency			
	λ_i^*,%	e_i^*,%	λ_i^{*s},%	sq_i^*	λ_i,%	e_i,%	λ_i^{sq},%	sq_i,%
1	37.33	3.23	53.39	3.04	54.05	38.30	45.95	32.56
2	53.46	2.07	52.71	0.74	59.80	5.76	40.20	3.87
3	25.43	2.02	1.08	0.42	59.80	3.63	40.20	2.44
4	0.86	0.47	1.58	0.38	54.05	2.47	45.95	2.10
5	36.71	0.22	24.13	0.24	59.80	1.94	40.20	1.30
6	0.87	0.12	0.86	0.21	54.05	0.77	45.95	0.65
7	1.90	0.08	41.06	0.11	59.80	0.54	40.20	0.37
8	52.56	0.08	41.65	0.05	54.05	0.43	45.95	0.37
9	0.98	0.02	23.51	0.04	54.05	0.34	45.95	0.29
10	1.95	0.01	1.10	0.03	59.80	0.34	40.20	0.24
11	0.96	0.01	24.11	0.03	54.05	0.28	45.95	0.22
12	1.96	0.00[a]	23.46	0.02	59.80	0.22	40.20	0.15
13	1.91	0.00[a]	52.01	0.01	59.80	0.09	40.20	0.00[b]
14	0.77	0.00[a]	0.87	0.01	54.05	0.08	45.95	0.00[b]
15	26.00	0.00[a]	51.59	0.00[b]	59.80	0.05	40.20	0.00[b]
16	0.77	0.00[a]	1.55	0.00[b]	54.05	0.02	45.95	0.00[b]

[a]0.00 < 0.01%. [b]0.00 < 0.001.

Table 3 Efficiency decomposition for ACOSS-FOUR

$i, j \rightarrow$ $\downarrow$	Global efficiency				Relative model efficiency			
$[N_{ij}]$	0.0242	0.0	12.4744	0.0	0.0027	0.0	65.2628	0.0
	2.5421	0.0	−0.0239	0.0	3.1580	0.0	0.2255	0.0
	−0.0001	0.0	0.1816	0.0	0.0414	0.0	0.7449	0.0
	0.0005	0.0	−0.0000	0.0	−0.1033	0.0	0.7688	0.0
$[D_{ij}]$	0.4510	0.0	232.2916	0.0	0.0096	0.0	229.8897	0.0
	9.0873	0.0	−0.0853	0.0	9.7632	0.0	0.6972	0.0
	−0.0026	0.0	3.3877	0.0	0.1281	0.0	2.3029	0.0
	0.0016	0.0	0.0002	0.0	−0.3636	0.0	2.7044	0.0

i = controller efficiency mode index (rows); j = structural mode index (columns).

$[D]$ matrices corresponds to the structural modal velocities and indicates zero-power expenditures for those states since there were no initial modal velocity disturbances. For relative model efficiency the (1, 3) elements of the decomposition matrices have the highest values indicating that the most power consumption occurs in the first controller mode corresponding to the displacement of the second (12th) structural mode (the third state variable). In fact, the values of the corresponding elements 65.26 and 229.90 are very close to the values of $S_C^M = 70.101$ and $S^R = 245.131$ given in Example 2. In addition, the structural mode efficiencies (e_j) were computed by using the coupled and decoupled structural mode efficiency definitions, Eqs. (38) and (39). The coupled structural mode efficiencies for the 12th and 11th structural modes were $e_j^* = 5.55, 0.64\%$, respectively, and $e_j = 27.41, 1.19\%$, respectively, for $j = 12, 11$. The decoupled efficiencies for the 11th and 12th structural modes were $e_j^* = 27.97, 5.37\%$, respectively, and $e_j = 32.32, 28.43\%$, respectively, for $j = 11, 12$. The expected efficiencies were found to be $E(e^*) = 6.22\%$ and $E(e) = 28.57\%$ by disregarding the off-block-diagonal modal coupling terms in the power matrices as per Eq. (40). These expected values are practically the same as the actual efficiencies given in Example 2. Note that in the case of decoupled structural mode efficiencies, the total contribution to power through off block-diagonal interactions is almost zero since the block-diagonal power consumption is practically the same as the respective total control power.

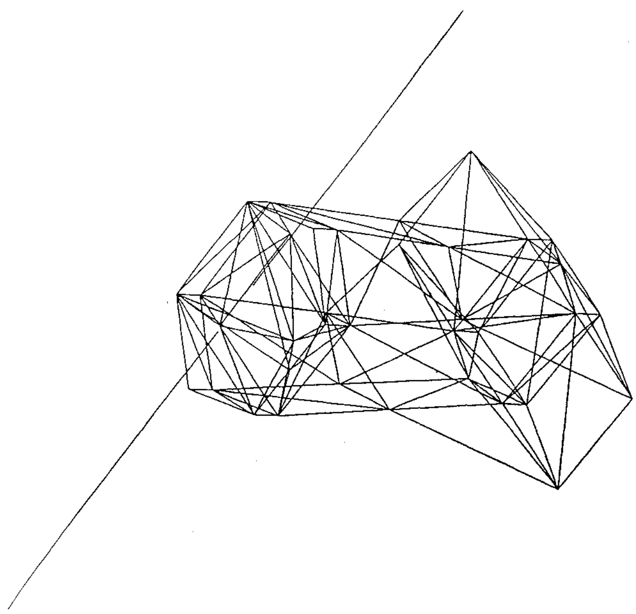

Fig. 4 ACOSS-SIX model.

ACOSS-SIX Structure (Model 2)

The CSDL structure shown in Fig. 4 was considered for efficiency analysis. The TM was obtained via NASTRAN for a 294-degree-of-freedom finite element model yielding a 588-state model. A unit initial displacement to the structure was given at node 37 at the top of the structure along each of the x, y, z directions. There were 21 actuators to choose from located at the bottom and top of the structure.

Efficiency Modes Analysis of ACOSS-6

Example 7. An eight-mode control design model was considered with four actuators $m = 4$ (inputs 18–21). The modes selected for the control design model were identified in the literature on this structure as important for the LOS errors. The chosen set was $n = \{12, 13, 17, 21, 22, 24, 28, 30\}$ with the natural frequencies $\omega_r = \{3.53179, 3.75708, 5.17415, 6.15384, 7.29905, 11.4184,$

Table 4 Structural mode efficiencies and controller mode efficiencies for ACOSS-SIX

Controller efficiency mode	Relative model efficiency, %		Structural mode	Structural mode efficiency e_j, %	
i	λ_i	e_i	j	Coupled	Decoupled
1	1.07	0.16	12, 13	0.20	1.06
2	0.64	0.14	22, 22	0.18	1.01
3	0.89	0.10	21, 24	0.14	0.99
4	0.32	0.07	24, 30	0.11	0.71
5	1.20	0.06	30, 17	0.02	0.69
6	0.32	0.06	28, 21	0.005	0.66
7	0.95	0.05	13, 12	0.004	0.35
8	0.71	0.02	17, 28	0.000	0.33
9	0.64	0.01			
10	0.33	0.00[a]			
11	1.13	0.00[a]	$e = 0.6571\%$, $E(e) = 0.6568\%$		
12	1.08	0.00[a]			
13	0.71	0.00[a]			
14	0.33	0.00[a]			
15	0.69	0.00[a]			
16	0.69	0.00[a]			

[a]0.00 < 0.01%.

21.6896, 24.6376} rad/s. The LQR control design had the weighting parameters $q = 1000$ and $r = 1$. The following results were obtained: $e^* = 0.176\%$, $e = 0.657\%$, $\mu = 3.72$, $\text{sq}^* = 563.168$, and $\text{sq} = 99.342\%$, corresponding to $S^* = 0.00113$, $S_C^M = 0.00421$, and $S^R = 0.6406$. The characteristic efficiencies λ_i and the controller mode (principal) efficiencies e_i are shown in Table 4 for the relative model efficiency. In addition, in Table 4, the coupled and decoupled structural mode efficiencies e_j of each structural mode are listed in order of their contributions. In particular, we again note that the expected efficiency obtained by disregarding the modal power coupling terms is practically the same as the actual efficiency. The analysis done above reflects the efficiency performance of the full 588th-state TM although the control-design model was 16th order and only 16th-order Lyapunov equations were solved to obtain these results. The results shown in this case should raise a concern about such low-efficiency control designs on realistic space structures although the designs are optimal for the CDPI of Eq. (42). The implication is that 99.34% of the control power lost by the control design has to be provided on board the spacecraft for no useful purpose, thus burdening the power subsystem with the likelihood of serious impacts on mission and spacecraft configuration. From an efficiency point the LQR optimality of the particular control-design model and the controller configuration should offer no comfort.

VII. Conclusions

The concept of controller efficiency modes signifying the quality of control power utilization has been developed. The controller modes have been presented in a way complementary to the familiar concept of structural modes, and together they determine the effectiveness of any structure-control design. The efficiency modal analysis yields a procedure to describe the internal power interactions in the SCS through spectral decomposition of the control power matrices. An important feature of the efficiency approach is that the performance of an entire TM can be judged on the basis of the control-design model alone without involving the truncated dynamics. The efficiency concepts have been illustrated on realistic structures. The efficiency modal analysis demonstrated for the control designs obtained by the LQR theory shows that highly inefficient control designs may result although they are regarded as optimal by the LQR theory. However, the optimality of an LQR design is subjective, and the theory merely serves as a means to obtain control gains. On the other hand, the definitions of power efficiencies of the control system constitute absolute nondimensional unique evaluation quantities. No subjectivity is involved in their definitions. The efficiency of all control systems regardless of the methodology by which they are designed can be analyzed by the approach presented in this paper and their merits can be established on a common ground. Finally, it is evident from the host of new information displayed by the cohabitation of the structural and controller efficiency modes in the SCS that there remains ample opportunity to exploit this new added dimension further to develop comprehensive efficiency-minded approaches to the structural control systems. In this regard, this paper is a prelude rather than an end in the concepts and definitions presented.

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